

DETERMINING NONSMOOTH FIRST ORDER TERMS FROM PARTIAL BOUNDARY MEASUREMENTS

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ABSTRACT. We extend results of Dos Santos Ferreira-Kenig-Sjöstrand-Uhlmann (arXiv:math.AP/0601466) to less smooth coefficients, and we show that measurements on part of the boundary for the magnetic Schrödinger operator determine uniquely the magnetic field related to a Hölder continuous potential. We give a similar result for determining a convection term. The proofs involve Carleman estimates, a smoothing procedure, and an extension of the Nakamura-Uhlmann pseudodifferential conjugation method to logarithmic Carleman weights.

1. INTRODUCTION

Let $\Omega \subseteq \mathbf{R}^n$, $n \geq 3$, be a bounded simply connected C^∞ domain with connected boundary. We will consider two inverse problems in Ω . The first is the inverse problem for the magnetic Schrödinger operator, defined by

$$H_{W,V} = \sum_{j=1}^n (D_j + W_j)^2 + V,$$

where $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, $W \in L^\infty(\Omega; \mathbf{C}^n)$, and $V \in L^n(\Omega)$. Provided that 0 is not a Dirichlet eigenvalue of $H_{W,V}$ in Ω , we define the Dirichlet-to-Neumann map (DN map) formally as the magnetic normal derivative

$$N_{W,V} : f \mapsto (\nabla + iW)u \cdot \nu|_{\partial\Omega},$$

where ν is the outer unit normal to $\partial\Omega$, and $u \in H^1(\Omega)$ solves the equation $H_{W,V}u = 0$ in Ω with $u|_{\partial\Omega} = f$. Using a weak formulation, $N_{W,V}$ is well defined as a bounded map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$, see also Section 2 below.

We consider $N_{W,V}$ as the boundary measurements related to the operator $H_{W,V}$, and the inverse problem is to determine the coefficients of $H_{W,V}$ from partial knowledge of $N_{W,V}$. There is gauge equivalence: one has $N_{W+\nabla p,V} = N_{W,V}$ when $p \in W^{1,\infty}(\Omega)$ and $p|_{\partial\Omega} = 0$. Thus one may only expect to recover the magnetic field dW and the electric potential V from the boundary measurements. Here dW is the 2-form $d(\sum_{j=1}^n W_j(x) dx_j)$.

Let $x_0 \in \mathbf{R}^n \setminus \overline{\text{ch}(\Omega)}$, where $\text{ch}(\Omega)$ is the convex hull of Ω . We define the front face of $\partial\Omega$ relative to x_0 by

$$F(x_0) = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) \leq 0\},$$

and we take \tilde{F} to be an open neighborhood of $F(x_0)$ in $\partial\Omega$. Also, let $W_{\text{tan}} = W - (W \cdot \nu)\nu$ be the tangential component on $\partial\Omega$ of a vector field W .

Our main result states that the magnetic field and electric potential are determined by measurements $N_{W,V}$ made on the particular subset \tilde{F} .

Theorem 1.1. Let $W_j \in C^\varepsilon(\overline{\Omega}; \mathbf{C}^n)$, $\varepsilon > 0$, and $V_j \in L^n(\Omega)$ for $j = 1, 2$. Also assume that 0 is not a Dirichlet eigenvalue of H_{W_j, V_j} in Ω . If

$$N_{W_1, V_1} f|_{\tilde{F}} = N_{W_2, V_2} f|_{\tilde{F}} \quad \text{for all } f \in H^{1/2}(\partial\Omega),$$

then $dW_1 = dW_2$ in Ω and $(W_1)_{\text{tan}} = (W_2)_{\text{tan}}$ on $\partial\Omega$. If further $V_j \in L^\infty(\Omega)$, then $V_1 = V_2$ in Ω .

This theorem was proved for $W \in C^2(\overline{\Omega})$ and $V \in L^\infty(\Omega)$ in [5], following the $W = 0$ case in [9] (see also [2]). These articles introduce important ideas, and the main tools are Carleman estimates, the construction of special solutions to the Schrödinger equation, and analytic microlocal analysis. The special solutions generalize the exponentially growing solutions in [20].

We will employ the methods of [5] and a smoothing argument from [15] to work with $W \in W^{1,n}(\Omega) \cap C^\varepsilon(\overline{\Omega})$. The assumption $W \in W^{1,n}(\Omega)$ ensures that the Carleman estimate construction gives special solutions in $H^1(\Omega)$. To remove this assumption, we will improve the construction of solutions by combining Carleman estimates with the pseudodifferential conjugation method from [13]. The conjugation method was given in [13] for linear Carleman weights, and we extend this method to logarithmic Carleman weights. This is a microlocal argument using some ideas from [6] and [9], and it gives the theorem for $W \in C^\varepsilon(\overline{\Omega})$. We also allow complex coefficients and give a more precise proof when Ω has complicated geometry.

Earlier work on the magnetic inverse problem for $n \geq 3$ has dealt with the case where the full DN map is known. The uniqueness question was solved in [19] for small W and in [12] for smooth W without smallness assumptions. The regularity of W was relaxed to C^1 in [21] and to Dini continuous in [18]. A constructive procedure for recovering dW and V from $N_{W,V}$ is given in [17]. The related inverse scattering problem has been studied in [7]. We also mention [1] and [22] which consider boundary determination and stability for magnetic Schrödinger operators, and [10] which proves a partial data result for the nonsmooth conductivity equation.

The other problem we consider is to determine a convection term from boundary measurements. If $W \in L^\infty(\Omega; \mathbf{R}^n)$ is a convection term and if $u \in H^1(\Omega)$ solves $(-\Delta + 2W \cdot \nabla)u = 0$ in Ω with $u|_{\partial\Omega} = f$, we define the related DN map formally as the normal derivative

$$N_W : f \mapsto \partial_\nu u|_{\partial\Omega}.$$

We remark that since W is real valued, 0 is not a Dirichlet eigenvalue of $-\Delta + 2W \cdot \nabla$ by the maximum principle. The map N_W is again bounded from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. The inverse problem of determining W from N_W can be reduced to the case of the magnetic Schrödinger equation, and we obtain the following result as a corollary to Theorem 1.1.

Theorem 1.2. Let $W_j \in C^\varepsilon(\overline{\Omega}; \mathbf{R}^n)$, $\varepsilon > 0$, and $\nabla \cdot W_j \in L^\infty(\Omega)$ for $j = 1, 2$. If

$$N_{W_1} f|_{\tilde{F}} = N_{W_2} f|_{\tilde{F}} \quad \text{for all } f \in H^{1/2}(\partial\Omega),$$

then $W_1 = W_2$ in Ω .

Earlier work on the convection term problem for $n \geq 3$ includes [3] which solves the full data problem for smooth W , and [18] which gives the same result for Lipschitz continuous W .

The paper is organized as follows. In Section 2 we fix some notation and preliminaries. Section 3 contains the facts on first order elliptic equations which will be needed for the pseudodifferential conjugation method. The Carleman estimates are in Section 4, and in Section 5 we establish the pseudodifferential conjugation method for logarithmic Carleman weights. Section 6 gives the construction of special solutions to the magnetic Schrödinger equation, and Theorems 1.1 and 1.2 are proved in Sections 7 and 8, respectively.

2. PRELIMINARIES

First we fix some notation. We will always take Ω to be a bounded simply connected domain in \mathbf{R}^n , $n \geq 3$, with connected C^∞ boundary. Recall that Ω is simply connected if it is connected and has trivial fundamental group (hence, the first de Rham cohomology vanishes). Let ν be the outer unit normal of Ω , and $\partial_\nu u = \nabla u \cdot \nu$ the normal derivative of u on $\partial\Omega$.

We write $C(\overline{\Omega})$, $C^\varepsilon(\overline{\Omega})$, and $C^\infty(\overline{\Omega})$ for the sets of complex valued functions which are continuous, ε -Hölder continuous, and infinitely differentiable, respectively, in $\overline{\Omega}$. The corresponding spaces of compactly supported functions in Ω are $C_c(\Omega)$, $C_c^\varepsilon(\Omega)$, and $C_c^\infty(\Omega)$. The notation $C(\overline{\Omega}; \mathbf{C}^n)$ denotes the space of n -dimensional vector fields whose components are in $C(\overline{\Omega})$, similarly for $C^\varepsilon(\overline{\Omega}; \mathbf{C}^n)$, $L^p(\Omega; \mathbf{C}^n)$ etc. For real-valued vector fields we write $C^\varepsilon(\overline{\Omega}; \mathbf{R}^n)$ etc.

If $u, v \in L^2(\Omega)$ we write $(u|v) = \int_\Omega u \bar{v} dx$, and also $(u|v) = \int_\Omega u \cdot \bar{v} dx$ when $u, v \in L^2(\Omega; \mathbf{C}^n)$. The $L^2(\Omega)$ norm is denoted by $\|u\| = (u|u)^{1/2}$. If $f, g \in L^2(\partial\Omega)$ let $(f|g)_{\partial\Omega} = \int_{\partial\Omega} f \bar{g} dx$. We will use the same notations for the pairing of distributions and elements in the dual space.

Denote by $H^s(\Omega)$ the L^2 Sobolev spaces in Ω , and by $W^{1,p}(\Omega)$ the usual L^p Sobolev spaces. We will need the following well-known results concerning multiplication in Sobolev spaces.

Lemma 2.1. If $a \in L^n(\Omega)$, $n \geq 3$, then $u \mapsto au$ maps $H^1(\Omega)$ to $L^2(\Omega)$ and

$$\|au\|_{L^2(\Omega)} \leq C \|a\|_{L^n(\Omega)} \|u\|_{H^1(\Omega)}. \quad (1)$$

If furthermore $a \in W^{1,n} \cap L^\infty(\Omega)$ then $u \mapsto au$ is bounded on $H^s(\Omega)$ for $-1 \leq s \leq 1$, and we have

$$\|au\|_{H^s(\Omega)} \leq C \|a\|_{W^{1,n} \cap L^\infty(\Omega)} \|u\|_{H^s(\Omega)}. \quad (2)$$

Proof. The first statement follows from the Hölder inequality and Sobolev embedding. The Leibniz rule and (1) give (2) for $s = 1$, the case $s = -1$ follows by duality, and the full result is obtained by interpolation. \square

Next we note that when W is Hölder continuous, one may always assume $\nabla \cdot W = 0$ after a gauge transformation which does not alter $N_{W,V}$ or the Hölder continuity of W .

Lemma 2.2. If $W \in C^\varepsilon(\overline{\Omega}; \mathbf{C}^n)$, then there exists $p \in C^{1+\varepsilon}(\overline{\Omega})$ such that $\nabla \cdot (W + \nabla p) = 0$ and $p|_{\partial\Omega} = 0$.

Proof. It is enough to choose p to be the solution of $\Delta p = -\nabla \cdot W$ with $p|_{\partial\Omega} = 0$, and this Dirichlet problem has a $C^{1+\varepsilon}$ solution by [8, Section 8.11]. \square

We proceed to give a precise definition of $N_{W,V}$. Let

$$H_\Delta(\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$$

with norm $\|u\|_{H_\Delta(\Omega)} = \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$. This is a Banach space and $C^\infty(\overline{\Omega})$ is a dense set. The trace operator is bounded on $H_\Delta(\Omega)$ with values in $H^{1/2}(\partial\Omega)$, and also $u \mapsto \partial_\nu u$ is bounded on $H_\Delta(\Omega)$ with values in $H^{-1/2}(\partial\Omega)$. This follows by writing $u \in H_\Delta(\Omega)$ as $u = u_0 + u_1$ where $\Delta u_0 = 0$ with $u_0|_{\partial\Omega} = u|_{\partial\Omega}$, and $\Delta u_1 = \Delta u$ with $u_1|_{\partial\Omega} = 0$. If $\nabla \cdot W = 0$ then for $f \in H^{1/2}(\partial\Omega)$, the solution u to $H_{W,V}u = 0$ with boundary value f is in $H_\Delta(\Omega)$. Therefore, the DN map $N_{W,V} : f \mapsto (\nabla + iW)u \cdot \nu|_{\partial\Omega}$ is bounded from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

Finally, we will need the Green identity in the following form.

Lemma 2.3. Let $W \in C(\overline{\Omega}; \mathbf{C}^n)$ with $\nabla \cdot W = 0$, and let $V \in L^n(\Omega)$. If $u \in H_\Delta(\Omega) \cap H_0^1(\Omega)$ and $v \in H_\Delta(\Omega)$, then

$$(H_{W,V}u|v) - (u|H_{\bar{W},\bar{V}}v) = -(\partial_\nu u|v)_{\partial\Omega}.$$

Proof. Let $u_j, v_j \in C^\infty(\overline{\Omega})$ with $u_j \rightarrow u$, $v_j \rightarrow v$ in $H_\Delta(\Omega)$. Then

$$(H_{W,V}u_j|v_j) - (u_j|H_{\bar{W},\bar{V}}v_j) = (u_j|\partial_\nu v_j)_{\partial\Omega} - (\partial_\nu u_j|v_j)_{\partial\Omega} - 2i((W \cdot \nu)u_j|v_j)_{\partial\Omega}.$$

The claim follows by taking limits. \square

3. ELLIPTIC EQUATIONS OF FIRST ORDER

To extend the Nakamura-Uhlmann pseudodifferential conjugation method to logarithmic Carleman weights, we will need to solve first order elliptic equations with variable coefficients in domains in $T^*\mathbf{R}^n$. This is because the symbols of the conjugating operators arise as solutions to such equations. The main result in this section is Proposition 3.4, which shows the solvability of a first order equation related to the logarithmic Carleman weight.

The standard reference for the following facts is [6]. See also [9] for the specific case of limiting Carleman weights, which will be the case of interest for us. Thus, let $\varphi = \varphi(x)$ be a smooth real function in an open set $V \subseteq \mathbf{R}^n$ with $\nabla\varphi \neq 0$ in V , and write $P = e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}}$. The semiclassical Weyl symbol of P is $p = a + ib$, where

$$a(x, \xi) = \xi^2 - (\nabla\varphi)^2, \quad b(x, \xi) = 2\nabla\varphi \cdot \xi. \quad (3)$$

We say that φ is a limiting Carleman weight (for the Laplacian) if the Poisson bracket $\{a, b\} = \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b$ satisfies

$$\{a, b\} = 0 \quad \text{when } a = b = 0. \quad (4)$$

This implies that $\{a, b\} = ca + db$ for some smooth c, d , and then on the set $\Sigma = \{a = b = 0\} \subseteq T^*V$ the Hamilton vector fields satisfy $[H_a, H_b] = cH_a + dH_b$. Recall that the Hamilton vector field of f is $H_f = \nabla_\xi f \cdot \nabla_x - \nabla_x f \cdot \nabla_\xi$.

It follows that Σ is an involutive manifold of codimension 2, and the Frobenius theorem states that at each point of Σ there are local coordinates in which $H_p = H_a + iH_b$ becomes an elliptic operator $a_1(y)\frac{\partial}{\partial y_1} + a_2(y)\frac{\partial}{\partial y_2}$. This implies local solvability of $H_p u = f$.

For our purposes local solvability is not enough, since we will need to solve a related equation in a full neighborhood of Σ . To do this we follow [6] and find a smooth function m satisfying $\{ma, mb\} = 0$ in a neighborhood of Σ , so that H_{ma} and H_{mb} commute near Σ and in some new coordinates H_{mp} becomes $\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}$.

The first lemma states that in the present case where a and b come from a limiting Carleman weight, there is an explicit choice for m .

Lemma 3.1. If $m = |\nabla\varphi(x)|^{-2}$, then $\{ma, mb\} = 0$ in T^*V .

Proof. From (3) and (4) one gets (see [9])

$$\{a, b\} = 4(\varphi''\xi \cdot \xi + \varphi''\nabla\varphi \cdot \nabla\varphi) = 4(c_1(x)a + (l_1(x) \cdot \xi)b) \quad (5)$$

for some c_1 and l_1 . Setting $\xi = 0$ and $\xi = \nabla\varphi$ gives $c_1 = -\frac{\varphi''\nabla\varphi \cdot \nabla\varphi}{(\nabla\varphi)^2}$ and $l_1 \cdot \nabla\varphi = -c_1$. By taking the terms in (5) which are of second order in ξ it follows that $(\varphi'' - c_1I - 2l_1(\nabla\varphi)^t)\xi \cdot \xi = 0$, which implies

$$\varphi'' - c_1I - l_1(\nabla\varphi)^t - \nabla\varphi(l_1)^t = 0.$$

Applying this matrix to $\nabla\varphi$ gives $l_1 = \frac{\varphi''\nabla\varphi}{(\nabla\varphi)^2}$.

Now $\nabla m = -2ml_1$, and

$$\begin{aligned} \{ma, mb\} &= \nabla_\xi(ma) \cdot \nabla_x(mb) - \nabla_x(ma) \cdot \nabla_\xi(mb) \\ &= m^2\{a, b\} + mb\nabla_\xi a \cdot \nabla_x m - ma\nabla_x m \cdot \nabla_\xi b \\ &= m^2(\{a, b\} - 4bl_1 \cdot \xi + 4al_1 \cdot \nabla\varphi) = 0. \end{aligned}$$

□

We now specialize to the weight $\varphi(x) = \log|x|$, and we will compute an explicit change of coordinates near Σ which makes H_{mp} into a $\bar{\partial}$ operator near Σ . In fact, the change of coordinates will consist of finding a codimension 2 manifold in a neighborhood of Σ which is transversal to the flows of H_{ma} and H_{mb} . We then need to check that the flows originating from this manifold cover a full neighborhood of Σ in the cotangent space.

Consider a truncated cone $V_0 = \{x \in \mathbf{R}^n : x_n > c|x|, c < |x| < c^{-1}\}$ for small $c > 0$. In suitable coordinates, V_0 will contain $\bar{\Omega}$. We will work in a fixed neighborhood

$$V = V(\delta) = \{x \in \mathbf{R}^n : x_n > (c - \delta)|x|, c - \delta < |x| < c^{-1} + \delta\}$$

with $\delta > 0$ small. Now Σ is given by $\{(x, \xi) \in T^*V : x \cdot \xi = 0, |x||\xi| = 1\}$, and we define a neighborhood

$$U = U(\delta) = \{(x, \xi) \in T^*V : |x \cdot \xi| < \delta, 1 - \delta < |x||\xi| < 1 + \delta\}.$$

The first two new coordinates will be

$$y_1 = x \cdot \xi, \quad y_2 = |x||\xi|,$$

so Σ is given by $\{y_1 = 0, y_2 = 1\}$.

We have $m = |x|^2$ and $ma = |x|^2|\xi|^2 - 1$, $mb = 2x \cdot \xi$, and the Hamilton vector fields are

$$H_{ma} = 2(|x|^2\xi \cdot \nabla_x - |\xi|^2x \cdot \nabla_\xi), \quad H_{mb} = 2(x \cdot \nabla_x - \xi \cdot \nabla_\xi).$$

These are smooth in $T^*\mathbf{R}^n$, and it is possible to compute the flows explicitly. If x and ξ are linearly independent, then the flows starting from (x, ξ) are given by

$$\begin{aligned} \theta_{ma}(s, (x, \xi)) &= (e^{2y_1s}|x|((\cos 2\sqrt{y_2^2 - y_1^2}s)\hat{x} + (\sin 2\sqrt{y_2^2 - y_1^2}s)J\hat{x}), \\ &\quad e^{-2y_1s}|\xi|((\sin 2\sqrt{y_2^2 - y_1^2}s)J\hat{\xi} + (\cos 2\sqrt{y_2^2 - y_1^2}s)\hat{\xi})), \\ \theta_{mb}(t, (x, \xi)) &= (e^{2t}x, e^{-2t}\xi), \end{aligned}$$

where $\hat{z} = \frac{z}{|z|}$, and for z in the oriented 2-plane $T = \text{span}(x, \xi)$, Jz is defined by

$$Jz = -(z \cdot \tilde{\xi})\hat{x} + (z \cdot \hat{x})\tilde{\xi}$$

where $\tilde{\xi} = \frac{\xi - (\xi \cdot \hat{x})\hat{x}}{|\xi - (\xi \cdot \hat{x})\hat{x}|}$. Thus, Jz is the unique vector in T for which $|Jz| = |z|$ and (z, Jz) is a positive orthogonal basis of T .

The leaf $\Gamma_{x,\xi}$ through $(x, \xi) \in T^*V$ is the set of all points in T^*V which can be reached from (x, ξ) by the flows of H_{ma} and H_{mb} .

Lemma 3.2. If $(x, \xi) \in U$ then $\Gamma_{x,\xi} \subseteq U$ and

$$\Gamma_{x,\xi} = \{(z, \eta) \in T^*V : z \cdot \eta = y_1, |z||\eta| = y_2, \text{span}(z, \eta) = \text{span}(x, \xi)\}.$$

Proof. Denote the set on the right by $L_{x,\xi}$. It is easy to see that $\Gamma_{x,\xi}$ is contained in $L_{x,\xi}$ by checking that $z \cdot \eta$, $|z||\eta|$, and $\text{span}(z, \eta)$ are constant along the flows. This also implies $\Gamma_{x,\xi} \subseteq U$.

For the converse, note that since $x \in V$ the plane $T = \text{span}(x, \xi)$ does not lie in $\{x_n = 0\}$, and we may define

$$\begin{aligned} w &= w(x, \xi) = \frac{\text{proj}_T e_n}{|\text{proj}_T e_n|}, \\ \zeta &= \zeta(x, \xi) = y_1 w + \sqrt{y_2^2 - y_1^2} Jw. \end{aligned}$$

It follows that $(w, \zeta) \in L_{x,\xi}$. We claim that $(x, \xi) = \theta_{mb}(t, \theta_{ma}(s, (w, \zeta)))$, provided that

$$\begin{aligned} s &= \frac{1}{2\sqrt{y_2^2 - y_1^2}} \angle(\hat{x} \cdot (w + iJw)), \\ t &= \frac{1}{2} \log |x| - y_1 s, \end{aligned}$$

where we define $\angle(e^{i\alpha}) = \alpha$ for $-\pi < \alpha < \pi$. In fact we have

$$(\cos 2\sqrt{y_2^2 - y_1^2}s)w + (\sin 2\sqrt{y_2^2 - y_1^2}s)Jw = (\hat{x} \cdot w)w + (\hat{x} \cdot Jw)Jw = \hat{x}$$

and so $\theta_{ma}(s, (w, \zeta)) = (e^{2y_1s}\hat{x}, \cdot)$ and $\theta_{mb}(t, \theta_{ma}(s, (w, \zeta))) = (x, \cdot)$. The claim follows since any two points in $L_{x,\xi}$ whose first components are the same must be identical. The construction also guarantees that the flow from (w, ζ) to (x, ξ) stays in T^*V .

Now, if $(z, \eta) \in L_{x, \xi}$, then $w(z, \eta) = w(x, \xi)$ and $\zeta(z, \eta) = \zeta(x, \xi)$. This shows that one may reach both (z, η) and (x, ξ) by flows starting from the same point (w, ζ) , which implies $(z, \eta) \in \Gamma_{x, \xi}$. \square

Since the leaves are given by the points where y_1, y_2 , and the plane $\text{span}(x, \xi)$ are constant, we take the next new variable to be this oriented plane. More precisely, we take (y_3, \dots, y_{2n-2}) to be local coordinates corresponding to $\text{span}(x, \xi)$ on the Grassmannian $G(2, n)$, which consists of oriented two-planes in \mathbf{R}^n . There is a single chart which achieves this, due to the fact that $\text{span}(x, \xi)$ does not lie in $\{x_n = 0\}$: one may apply an oriented version of Plücker coordinates, or if $n = 3$ it is sufficient to use the identification $G(2, 3) = S^2$ and stereographic projection.

The final coordinates will be the flow variables $y_{2n-1} = s$ and $y_{2n} = t$, where s and t were given in the proof of Lemma 3.2. The codimension 2 manifold transversal to the flows will then be $\{y_{2n-1} = y_{2n} = 0\}$. We have arrived at the desired change of coordinates.

Lemma 3.3. The map $\Phi : (x, \xi) \mapsto y$ defined in the discussion above is smooth and injective on $U(\delta)$ for δ small, and it is a diffeomorphism onto its image in \mathbf{R}^{2n} . In the new coordinates H_{mp} becomes $\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}$.

Proof. If $\Phi(x, \xi) = \Phi(x', \xi')$ where $(x, \xi), (x', \xi') \in U$, then Lemma 3.2 implies that (x', ξ') is on the leaf through (x, ξ) . Also, the vectors w, ζ and the coordinates s, t in Lemma 3.2 are the same whether they are computed from (x, ξ) or (x', ξ') . It follows that $(x, \xi) = \theta_{mb}(t, \theta_{ma}(s, (w, \zeta))) = (x', \xi')$, which shows that Φ is injective. Since Φ is smooth, it is a diffeomorphism onto its image. Further, H_{mp} becomes $\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}$ because the flows of H_{ma} and H_{mb} commute. \square

Note that $H_{mp} = mH_p$ on Σ , so the following result states in particular that one may solve $H_p u = f$ on Σ .

Proposition 3.4. If $f(x, \xi) \in C^\infty(\overline{U(2\delta)})$ for δ small, then the equation

$$H_{mp}u = mf \quad \text{in } U(\delta)$$

has a solution $u \in C^\infty(U(\delta))$ satisfying for all $k > 0$

$$|\partial_x^\alpha \partial_\xi^\beta u(x, \xi)| \leq C_k \|f\|_{W^{k, \infty}(U(2\delta))}, \quad (x, \xi) \in U(\delta)$$

whenever $|\alpha| + |\beta| \leq k$.

Proof. Writing $u = \tilde{u} \circ \Phi$, it is enough to solve

$$(\partial_{y_{2n-1}} + i\partial_{y_{2n}})\tilde{u} = (\chi mf) \circ \Phi^{-1} \quad \text{on } \Phi(U(2\delta))$$

where $\chi \in C_c^\infty(U(2\delta))$ and $\chi = 1$ near $U(\delta)$. This may be solved using the Cauchy transform $(\partial_{y_{2n-1}} + i\partial_{y_{2n}})^{-1}$, and the norm estimates are an immediate consequence. \square

4. CARLEMAN ESTIMATES

In this section we will first recall the Carleman estimates for $H_{W,V}$ in [5], and we note that the shifted estimate is valid when $W \in W^{1,n} \cap L^\infty(\Omega; \mathbf{C}^n)$. We then use the Carleman estimate to solve an equation involving a conjugated version of $H_{W,V}$.

The Carleman estimate in [5] is an estimate for the magnetic Schrödinger equation proved for limiting Carleman weights. Denote by $\tilde{\Omega} \subseteq \mathbf{R}^n$ an open set such that $\bar{\Omega} \subseteq \tilde{\Omega}$, and recall that φ is a limiting Carleman weight in $\tilde{\Omega}$ if (3) and (4) are satisfied in $T^*\tilde{\Omega}$. Later we will restrict ourselves to the particular logarithmic weight. Below we write $\partial\Omega_\pm = \{x \in \partial\Omega : \pm \partial_\nu \varphi(x) \geq 0\}$, and $A \lesssim B$ if $A \leq CB$ where $C > 0$ is a constant independent of h and ε . Recall that $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$.

Proposition 4.1. Let φ be a limiting Carleman weight on $\tilde{\Omega}$. Suppose $W \in W^{1,n} \cap L^\infty(\Omega; \mathbf{C}^n)$ and $V \in L^n(\Omega)$. Then for $u \in H_\Delta \cap H_0^1(\Omega)$ and h small, we have the Carleman estimate

$$\begin{aligned} & -h(\partial_\nu \varphi e^{\frac{\varphi}{h}} \partial_\nu u |e^{\frac{\varphi}{h}} \partial_\nu u)_{\partial\Omega_-} + \|e^{\frac{\varphi}{h}} u\|^2 + \|e^{\frac{\varphi}{h}} h \nabla u\|^2 \\ & \lesssim h^2 \|e^{\frac{\varphi}{h}} H_{W,V} u\|^2 + h(\partial_\nu \varphi e^{\frac{\varphi}{h}} \partial_\nu u |e^{\frac{\varphi}{h}} \partial_\nu u)_{\partial\Omega_+}. \end{aligned} \quad (6)$$

Proof. By introducing $v = e^{\frac{\varphi}{h}} u$ it follows that (6) is equivalent to the a priori estimate

$$\begin{aligned} & -h(\partial_\nu \varphi \partial_\nu v | \partial_\nu v)_{\partial\Omega_-} + \|v\|^2 + \|h \nabla v\|^2 \\ & \lesssim h^2 \|e^{\frac{\varphi}{h}} H_{W,V} e^{-\frac{\varphi}{h}} v\|^2 + h(\partial_\nu \varphi \partial_\nu v | \partial_\nu v)_{\partial\Omega_+} \end{aligned} \quad (7)$$

for the conjugated operator $e^{\frac{\varphi}{h}} H_{W,V} e^{-\frac{\varphi}{h}}$. To prove (7) the idea is to work with the convexified weight

$$\tilde{\varphi}(x) = \varphi(x) + h \frac{\varphi(x)^2}{2\varepsilon}$$

and first obtain the estimate for $e^{\frac{\tilde{\varphi}}{h}} H_{W,V} e^{-\frac{\tilde{\varphi}}{h}}$. This operator can be split into a self-adjoint term \tilde{P} , a skew-adjoint term $i\tilde{Q}$, and a remainder term \tilde{R} ,

$$e^{\frac{\tilde{\varphi}}{h}} H_{W,V} e^{-\frac{\tilde{\varphi}}{h}} = h^{-2}(\tilde{P} + i\tilde{Q} + \tilde{R}),$$

where

$$\begin{aligned} \tilde{P} &= -h^2 \Delta - (\nabla \tilde{\varphi})^2, \\ \tilde{Q} &= hD \circ \nabla \tilde{\varphi} + \nabla \tilde{\varphi} \circ hD, \\ \tilde{R} &= 2hW \cdot (hD + i\nabla \tilde{\varphi}) + h^2(W^2 + D \cdot W + V). \end{aligned}$$

We then have from [5, Eq. (2.12)] the estimate

$$\frac{h^2}{\varepsilon} (\|v\|^2 + \|h \nabla v\|^2) \lesssim \|(\tilde{P} + i\tilde{Q})v\|^2 + h^3 (\partial_\nu \tilde{\varphi} \partial_\nu v | \partial_\nu v)_{\partial\Omega} \quad (8)$$

for some sufficiently small $\varepsilon > 0$. By the assumptions on W and V , it follows from (1) that

$$\begin{aligned} \|\tilde{R}v\|^2 &\lesssim h^2(\|W\|_{L^\infty(\Omega)}^2 + \|W^2 + D \cdot W + V\|_{L^n(\Omega)}^2)(\|v\|^2 + \|h\nabla v\|^2) \\ &\lesssim h^2(\|v\|^2 + \|h\nabla v\|^2). \end{aligned}$$

Hence by choosing $\varepsilon > 0$ sufficiently small, the term $\|\tilde{R}v\|^2$ can be absorbed in the left hand side of (8) and consequently

$$\frac{h^2}{\varepsilon}(\|v\|^2 + \|h\nabla v\|^2) \lesssim h^4 \|e^{\frac{\tilde{\varphi}}{h}} H_{W,V} e^{-\frac{\tilde{\varphi}}{h}} v\|^2 + h^3 (\partial_\nu \tilde{\varphi} \partial_\nu v | \partial_\nu v)_{\partial\Omega},$$

from which (7) can be derived as in [5]. \square

Next we show that the Carleman estimate (6) can be shifted to a lower Sobolev index. For this we will use the semiclassical Sobolev spaces $H_{\text{scl}}^s(\mathbf{R}^n)$ with norm $\|f\|_{H_{\text{scl}}^s(\mathbf{R}^n)} = \|\langle hD \rangle^s f\|_{L^2(\mathbf{R}^n)}$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Proposition 4.2. Let φ be a limiting Carleman weight, and suppose that $W \in W^{1,n} \cap L^\infty(\Omega; \mathbf{C}^n)$ and $V \in L^n(\Omega)$. Then for $v \in C_c^\infty(\Omega)$ we have the estimate

$$\|v\|_{L^2(\mathbf{R}^n)}^2 \lesssim h^2 \|e^{\frac{\tilde{\varphi}}{h}} H_{W,V} e^{-\frac{\tilde{\varphi}}{h}} v\|_{H_{\text{scl}}^{-1}(\mathbf{R}^n)}^2. \quad (9)$$

Proof. Using the notation in Proposition 4.1 we have from [5, Proposition 2.4] the estimate

$$\frac{h^2}{\varepsilon} \|v\|_{H_{\text{scl}}^1(\mathbf{R}^n)}^2 \lesssim \|(\tilde{P} + i\tilde{Q}) \langle hD \rangle v\|_{H_{\text{scl}}^{-1}(\mathbf{R}^n)}^2,$$

for $v \in C_c^\infty(\hat{\Omega})$, where $\hat{\Omega}$ is open and $\bar{\Omega} \subseteq \hat{\Omega}$ and $\bar{\hat{\Omega}} \subseteq \tilde{\Omega}$. Suppose we have extended W, V to \mathbf{R}^n . The semiclassical counterparts of (1) and (2) are

$$\begin{aligned} h\|au\|_{L^2(\mathbf{R}^n)} &\lesssim \|a\|_{L^n(\mathbf{R}^n)} \|u\|_{H_{\text{scl}}^1(\mathbf{R}^n)}, \\ \|au\|_{H_{\text{scl}}^1(\mathbf{R}^n)} &\lesssim \|a\|_{W^{1,n} \cap L^\infty(\mathbf{R}^n)} \|u\|_{H_{\text{scl}}^1(\mathbf{R}^n)}, \end{aligned}$$

and the corresponding dual estimates yield

$$\|\tilde{R}w\|_{H_{\text{scl}}^{-1}(\mathbf{R}^n)} \lesssim h\|w\|_{L^2(\mathbf{R}^n)}.$$

Hence for $\varepsilon > 0$ sufficiently small we have

$$\frac{h^2}{\varepsilon} \|v\|_{H_{\text{scl}}^1(\mathbf{R}^n)}^2 \lesssim h^4 \|e^{\frac{\tilde{\varphi}}{h}} H_{W,V} e^{-\frac{\tilde{\varphi}}{h}} \langle hD \rangle v\|_{H_{\text{scl}}^{-1}(\mathbf{R}^n)}^2,$$

from which (9) can be derived as in [5]. \square

As a consequence of the estimate (9) and the Hahn-Banach theorem, we have the following solvability result.

Proposition 4.3. Let φ be a limiting Carleman weight, and let $W \in W^{1,n} \cap L^\infty(\Omega; \mathbf{C}^n)$ and $V \in L^n(\Omega)$. If h is sufficiently small, then for any $f \in L^2(\Omega)$ the equation

$$e^{\frac{\varphi}{h}} H_{W,V} e^{-\frac{\varphi}{h}} u = f \quad (10)$$

has a solution $u \in H^1(\Omega)$ with

$$\|u\|_{L^2(\Omega)} + \|hDu\|_{L^2(\Omega)} \lesssim h\|f\|_{L^2(\Omega)}. \quad (11)$$

5. PSEUDODIFFERENTIAL CONJUGATION

In this section we will prove Proposition 4.3 in the case of uniformly continuous coefficients. This will follow by extending the pseudodifferential conjugation technique introduced in [13] (see also [14]) to logarithmic Carleman weights

$$\varphi(x) = \log|x - x_0|, \quad x_0 \notin \overline{\Omega}. \quad (12)$$

The main result is the following.

Proposition 5.1. Let $\varphi(x)$ be as in (12), and assume that $W \in C(\overline{\Omega}; \mathbf{C}^n)$ and $\nabla \cdot W, V \in L^n(\Omega; \mathbf{C})$. If h is sufficiently small, then for any $f \in L^2(\Omega)$ the equation (10) has a solution $u \in H^1(\Omega)$ satisfying (11).

To prove this, we start by choosing $\tilde{\Omega}$ to be an open set containing $\overline{\Omega}$ with $x_0 \notin \tilde{\Omega}$, and we take coordinates so that $x_0 = 0$ and $\tilde{\Omega} \subseteq \{x_n > c|x|, c < |x| < c^{-1}\}$ for c small. Note that $H_{W,V} = -\Delta + 2W \cdot D + \tilde{V}$ with $\tilde{V} \in L^n(\tilde{\Omega})$. We ignore the zero order term for the moment and consider the conjugated operator

$$e^{\frac{\varphi}{h}}((hD)^2 + 2hW \cdot hD)e^{-\frac{\varphi}{h}} = P + hQ$$

where $P = (hD + i\nabla\varphi)^2$ and $Q = 2W \cdot (hD + i\nabla\varphi)$.

If h is sufficiently small, Proposition 4.3 (in the case $W = V = 0$) implies that we may solve $Pu = f$ in $\tilde{\Omega}$ for any $f \in L^2(\tilde{\Omega})$, and the solution operator $P^{-1} : f \mapsto u$ is a linear map which satisfies

$$\|P^{-1}f\|_{L^2(\tilde{\Omega})} + \|hDP^{-1}f\|_{L^2(\tilde{\Omega})} \lesssim h^{-1}\|f\|_{L^2(\tilde{\Omega})}. \quad (13)$$

We would like to solve $(P + hQ)u = f$ in the same way. If $\|W\|_{L^\infty(\tilde{\Omega})}$ is small then $P + hQ$ is a small perturbation of P , and a solution is obtained from the Neumann series. If W is large a different method is needed. In this case we will find order 0 pseudodifferential operators which conjugate the operator into a small perturbation of P . We use standard classes of semiclassical symbols and operators.

Definition. If $0 \leq \sigma < 1/2$ and $m \in \mathbf{R}$, let S_σ^m be the space of all functions $a(x, \xi; h)$ where $x, \xi \in \mathbf{R}^n$ and $h \in (0, h_0]$, $h_0 \leq 1$, such that a is smooth in x and ξ and

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha\beta} h^{-\sigma|\alpha+\beta|} (1 + |\xi|^2)^{m/2}$$

for all α, β . If $a \in S_\sigma^m$ we define an operator $A = \text{Op}_h(a)$ by

$$Af(x) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{i(x-y) \cdot \xi} a\left(\frac{x+y}{2}, h\xi\right) f(y) d\xi dy.$$

Note that we define the operators using semiclassical Weyl quantization. Operators in S_σ^0 are bounded on L^2 with norm uniformly bounded in h , and the composition of two operators is again an operator in the same class. We assume familiarity with semiclassical calculus in what follows, for more details see [4] and also [17] where the result of this section was proved in the semiclassical setup for linear Carleman weights.

To deal with the nonsmooth coefficients we extend W to a vector field in $C_c(\tilde{\Omega}; \mathbf{C}^n)$ and \tilde{V} by zero, and we consider the mollifications

$$W^\sharp = W * \chi_\delta, \quad \tilde{V}^\sharp = \tilde{V} * \chi_\delta \quad (14)$$

where $\delta = h^\sigma$ and $0 < \sigma < 1/2$. Here $\chi_\delta(x) = \delta^{-n} \chi(x/\delta)$ is the usual mollifier with $\chi \in C_c^\infty(\mathbf{R}^n)$, $0 \leq \chi \leq 1$, and $\int \chi dx = 1$. We write $W^b = W - W^\sharp$ and $\tilde{V}^b = \tilde{V} - \tilde{V}^\sharp$ and note the estimates

$$\|\partial^\alpha W^\sharp\|_{L^\infty}, \|\partial^\alpha \tilde{V}^\sharp\|_{L^n} = O(h^{-\sigma|\alpha|}), \quad (15)$$

$$\|W^b\|_{L^\infty}, \|\tilde{V}^b\|_{L^n} = o(1) \quad (16)$$

as $h \rightarrow 0$.

Write $q^\sharp = 2W^\sharp \cdot (\xi + i\nabla\varphi) - hD \cdot W^\sharp$, so that $Q^\sharp = \text{Op}_h(q^\sharp) = 2W^\sharp \cdot (hD + i\nabla\varphi)$. The following lemma shows that one may use pseudodifferential operators to conjugate away the main part Q^\sharp of the first order term Q .

Lemma 5.2. There exist $c, \tilde{c}, r \in S_\sigma^0$ so that

$$(P + hQ^\sharp)C = \tilde{C}P + h^{2-2\sigma}R \quad \text{in } \Omega.$$

Both C and \tilde{C} are elliptic, in the sense that c and \tilde{c} are nonvanishing for small h .

Proof. Suppose c is any symbol in S_σ^0 which is equal to 1 outside a large ball in (x, ξ) . Then for some $r_0 \in S_\sigma^0$,

$$(P + hQ^\sharp)C = CP + h\text{Op}_h\left(\frac{1}{i}H_p c + q^\sharp c\right) + h^{2-2\sigma}R_0 \quad \text{in } \Omega. \quad (17)$$

We will choose c so that $\frac{1}{i}H_p c + q^\sharp c$ is of lower order. Trying $c = e^{i\phi}$ and using the notation in Section 3 and Proposition 3.4, we choose ϕ to be a solution of

$$H_{mp}\phi = -mq^\sharp \quad \text{in } U = U(\delta). \quad (18)$$

Since x and ξ are bounded in U , (15) gives

$$|\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \leq C_{\alpha\beta} h^{-\sigma|\alpha+\beta|}, \quad (x, \xi) \in U.$$

Let $\chi(x, \xi)$ be a smooth cutoff supported in $U(\delta)$ which is equal to 1 on $U(\delta/2)$. Then c is chosen to be

$$c = e^{i\chi\phi}.$$

It follows that $c \in S_\sigma^0$ and c is nonvanishing.

Since $H_{mp} = mH_p + pH_m$, we get by (18)

$$\begin{aligned} m\left(\frac{1}{i}H_p c + q^\sharp c\right) &= ((H_{mp}\chi)\phi + \chi H_{mp}\phi - pH_m(\chi\phi) + mq^\sharp)c \\ &= ((H_{mp}\chi)\phi + (1 - \chi)mq^\sharp - pH_m(\chi\phi))c. \end{aligned}$$

Let $\psi \in C_c^\infty(\tilde{\Omega})$ and $\psi = 1$ near $\bar{\Omega}$. Then $\psi(\frac{1}{i}H_p c + q^\sharp c) = lp$ where

$$l = \psi\left(\frac{1 - \chi}{p}q^\sharp + \frac{1}{mp}(H_{mp}\chi)\phi - \frac{1}{m}H_m(\chi\phi)\right)c$$

and $l \in h^{-\sigma}S_\sigma^0$. Then (17) gives

$$(P + hQ^\sharp)C = \tilde{C}P + h^{2-2\sigma}R \quad \text{in } \Omega$$

where $\tilde{c} = c + hl$ is of order 0 and elliptic, and $r \in S_\sigma^0$. \square

We may now prove the main result. The proof involves a number of cutoffs to restrict the functions to the set $\tilde{\Omega}$ where φ is smooth.

Proof. (of Proposition 5.1) Using the notation in this section, we want to solve

$$(P + hQ^\sharp + hQ^\flat + h^2\tilde{V}^\sharp + h^2\tilde{V}^\flat)u = h^2f \quad \text{in } \Omega \quad (19)$$

where $Q^\flat = 2W^\flat \cdot (hD + i\nabla\varphi)$. Let ψ_j ($j \geq 1$) be $C_c^\infty(\tilde{\Omega})$ functions with $\psi_1 = 1$ near $\overline{\Omega}$ and $\psi_{j+1} = 1$ near $\text{supp}(\psi_j)$. We try a solution

$$u = C\psi_2 P^{-1}\psi_3 \tilde{C}^{-1}\psi_4 v \quad (20)$$

for some $v \in L^2(\mathbf{R}^n)$. Here \tilde{C}^{-1} is the inverse of \tilde{C} on $L^2(\mathbf{R}^n)$, which exists for small h . Inserting (20) in (19) and using Lemma 5.2 gives

$$(M + R_1)v = h^2f \quad \text{in } \Omega \quad (21)$$

where $M = \tilde{C}P\psi_2 P^{-1}\psi_3 \tilde{C}^{-1}\psi_4$ and $\|\psi_1 R_1\|_{L^2 \rightarrow L^2} = o(1)$ as $h \rightarrow 0$. Here we have used (1), (13), (15), (16), and the identity $\partial_j C = C\partial_j + \text{Op}_h(\partial_{x_j} c)$.

We may write

$$M = \psi_2 + [\tilde{C}, \psi_2]\tilde{C}^{-1}\psi_4 + \tilde{C}[P, \psi_2]P^{-1}\psi_3 \tilde{C}^{-1}\psi_4.$$

Extending f by zero and modifying (21) slightly, we obtain an equation in \mathbf{R}^n of the form

$$(I + M_1 + \psi_1 R_1)v = h^2f \quad \text{in } \mathbf{R}^n \quad (22)$$

where $M_1 = \psi_1[\tilde{C}, \psi_2]\tilde{C}^{-1}\psi_4 + \psi_1\tilde{C}[P, \psi_2]P^{-1}\psi_3 \tilde{C}^{-1}\psi_4$. By the pseudolocal property $\|M_1\|_{L^2 \rightarrow L^2} = o(1)$ as $h \rightarrow 0$, so we obtain a solution $v \in L^2(\mathbf{R}^n)$ to (22) by Neumann series. Since $\psi_j = 1$ on $\overline{\Omega}$ the function v will satisfy (21) near $\overline{\Omega}$. It follows that (20) solves (19), and one obtains the desired norm estimates from (13). \square

6. CONSTRUCTION OF SOLUTIONS

We now give the construction of special solutions to $H_{W,V}u = 0$ in Ω . This proceeds as in [5], except that an additional smoothing argument is required. The solutions will have the form $u = e^{-\frac{\varphi}{h}}v$ where v is a WKB solution for the conjugated operator $e^{\frac{\varphi}{h}}H_{W,V}e^{-\frac{\varphi}{h}}$. Thus, u is of the form

$$u = e^{-\frac{\rho}{h}}(a + r) \quad (23)$$

where $\rho = \varphi + i\psi$ and ψ is a real valued phase function, a is an amplitude, and r is a correction term. By Lemma 2.2 we may assume that $\nabla \cdot W = 0$, and then

$$h^2 H_{W,V} = (hD)^2 + 2hW \cdot hD + h^2\tilde{V}$$

with $\tilde{V} \in L^n$, and inserting (23) into $h^2 H_{W,V}u = 0$ gives the equation

$$((hD + i\nabla\rho)^2 + 2hW \cdot (hD + i\nabla\rho) + h^2\tilde{V})(a + r) = 0.$$

Collecting like powers of h , u will be a solution provided that one has in Ω

$$(\nabla \rho)^2 = 0, \quad (24)$$

$$(\nabla \rho \cdot D + \nabla \rho \cdot W + \frac{1}{2i} \Delta \rho) a = 0, \quad (25)$$

$$e^{\frac{\rho}{h}} h^2 H_{W,V} e^{-\frac{\rho}{h}} r = -h^2 H_{W,V} a. \quad (26)$$

We fix $\varphi(x)$ defined by (12). Now (24) is an eikonal equation for ψ , which reads

$$(\nabla \psi)^2 = (\nabla \varphi)^2, \quad \nabla \varphi \cdot \nabla \psi = 0.$$

As shown in [9], a solution is given by

$$\psi(x) = \text{dist}_{S^{n-1}} \left(\frac{x - x_0}{|x - x_0|}, \omega \right) \quad (27)$$

where $\omega \in S^{n-1}$ is chosen so that ψ is smooth in $\tilde{\Omega}$.

We choose coordinates so that $x_0 = 0$ and $\tilde{\Omega} \subseteq \{x_n > 0\}$, and we take $\omega = e_1$. We also write $x = (x_1, r\theta)$ with $r > 0$ and $\theta \in S^{n-2}$, and consider the change of variables $\Psi : x \mapsto (z, \theta)$ where $z = x_1 + ir$ is a complex variable. Writing $\tilde{f} = f \circ \Psi^{-1}$, we get $\tilde{\rho} = \log z$, $(\nabla \rho)^\sim = \frac{1}{z}(e_1 + ie_r)$ where $e_r = (0, \theta)$, $(\Delta \rho)^\sim = -\frac{2(n-2)}{z(z-\bar{z})}$, and $\nabla \rho \cdot \nabla$ becomes $\frac{2}{z} \partial_{\bar{z}}$.

We see that (25) is a $\bar{\partial}$ -equation in the new coordinates, and the solution a will have in general the same regularity as W . On the other hand, in (26) one needs two derivatives of a on the right. To deal with this for $W \in C^\varepsilon(\tilde{\Omega}; \mathbf{C}^n)$ we use the same approximation procedure as in (14), now with $\sigma > 0$ small and W extended as a vector field in $C_c^\varepsilon(\tilde{\Omega}; \mathbf{C}^n)$. The solution will be taken of the form $u = e^{-\frac{\rho}{h}}(a + r)$, where ρ is as above and one has in Ω

$$(\nabla \rho \cdot D + \nabla \rho \cdot W^\sharp + \frac{1}{2i} \Delta \rho) a = 0, \quad (28)$$

$$e^{\frac{\rho}{h}} h^2 H_{W,V} e^{-\frac{\rho}{h}} r = -2ih(\nabla \rho \cdot W^\flat) a - h^2 H_{W,V} a. \quad (29)$$

To solve (28) we take $a = \tilde{a} \circ \Psi$, so \tilde{a} must satisfy

$$\left(\partial_{\bar{z}} + \frac{i}{2}(e_1 + ie_r) \cdot \tilde{W}^\sharp - \frac{n-2}{2(z-\bar{z})} \right) \tilde{a} = 0 \quad \text{in } \Psi(\Omega).$$

Here $\tilde{W}^\sharp = W^\sharp \circ \Psi^{-1}$. We try $\tilde{a} = (z - \bar{z})^{\frac{2-n}{2}} e^{i\Phi}$, and we get the equation

$$\partial_{\bar{z}} \Phi = -\frac{1}{2}(e_1 + ie_r) \cdot \tilde{W}^\sharp \quad \text{in } \Psi(\Omega). \quad (30)$$

It is now easy to solve (28).

Lemma 6.1. The equation (28) has a solution $a \in C^\infty(\bar{\Omega})$ which satisfies $\|\partial^\alpha a\|_{L^\infty(\Omega)} \leq C_\alpha h^{-\sigma|\alpha|}$.

Proof. We first take $\chi \in C_c^\infty(\tilde{\Omega})$ with $\chi = 1$ near Ω , and consider

$$\partial_{\bar{z}} \Phi = -\frac{1}{2} \tilde{\chi}(e_1 + ie_r) \cdot \tilde{W}^\sharp \quad \text{for } z \in \mathbf{C}$$

with $\tilde{\chi} = \chi \circ \Psi^{-1}$. This has the explicit solution

$$\Phi(z, \theta) = -\frac{1}{4\pi i} \int_{\mathbf{C}} \frac{1}{w} \tilde{\chi}(z - w, \theta) (e_1 + ie_r) \cdot \tilde{W}^\sharp(z - w, \theta) d\bar{w} \wedge dw. \quad (31)$$

Then $a = ((z - \bar{z})^{\frac{2-n}{2}} e^{i\Phi}) \circ \Psi$ will be a solution of (28) in Ω with the given norm bounds. \square

It remains to solve (29). The L^2 norm of the right hand side of (29) is $O(h^{1+\sigma\varepsilon})$ when σ is small enough, using the Hölder continuity of W . It follows from Proposition 5.1 that when h is small there is an $H^1(\Omega)$ solution r with $\|r\|_{L^2(\Omega)} = O(h^{\sigma\varepsilon})$, $\|\nabla r\|_{L^2(\Omega)} = O(h^{-1+\sigma\varepsilon})$.

We collect the results obtained in this argument.

Proposition 6.2. Let $W \in C^\varepsilon(\bar{\Omega}; \mathbf{C}^n)$, $\nabla \cdot W = 0$, and $V \in L^n(\Omega)$. Let φ, ψ be defined by (12) and (27) respectively. Then for h small there is an $H^1(\Omega)$ solution $u = e^{-\frac{1}{h}(\varphi+i\psi)}(a+r)$ of the equation $H_{W,V}u = 0$ in Ω , where a is given in Lemma 6.1. Also, one has the norm estimates

$$\begin{aligned} \|\partial^\alpha a\|_{L^\infty(\Omega)} &= O(h^{-\sigma|\alpha|}), \\ \|r\|_{L^2(\Omega)} + \|h\nabla r\|_{L^2(\Omega)} &= O(h^{\sigma\varepsilon}) \end{aligned}$$

where $\sigma > 0$ is small.

7. DETERMINING A MAGNETIC FIELD

In this section we prove Theorem 1.1 following the arguments given in [5]. Assume that $W_j \in C^\varepsilon(\Omega; \mathbf{C}^n)$, $\nabla \cdot W_j = 0$, $V_j \in L^n(\Omega)$, and that

$$N_{W_1, V_1} f|_{\tilde{F}} = N_{W_2, V_2} f|_{\tilde{F}} \quad \text{for all } f \in H^{1/2}(\partial\Omega). \quad (32)$$

Also assume that W_j are extended as vector fields in $C_c^\varepsilon(\tilde{\Omega}; \mathbf{C}^n)$.

We start by choosing u_1 and u_2 to be solutions of $H_{W_1, V_1} u_1 = 0$ and $H_{W_2, V_2} u_2 = 0$, of the form

$$\begin{aligned} u_1 &= e^{\frac{1}{h}(\varphi+i\psi)}(a_1 + r_1), \\ u_2 &= e^{\frac{1}{h}(-\varphi+i\psi)}(a_2 + r_2) \end{aligned}$$

where $\varphi(x)$ and $\psi(x)$ are defined by (12) and (27) respectively, a_j satisfies

$$\begin{aligned} (\nabla(\varphi + i\psi) \cdot D + \nabla(\varphi + i\psi) \cdot W_1^\sharp + \frac{1}{2i}\Delta(\varphi + i\psi))a_1 &= 0, \\ (\nabla(-\varphi + i\psi) \cdot D + \nabla(-\varphi + i\psi) \cdot \bar{W}_2^\sharp + \frac{1}{2i}\Delta(-\varphi + i\psi))a_2 &= 0, \end{aligned}$$

and $\|\partial^\alpha a_j\|_{L^\infty(\Omega)} = O(h^{-\sigma|\alpha|})$ for $\sigma > 0$ small, and finally $\|r_j\| = O(h^{\sigma\varepsilon})$, $\|\nabla r_j\| = O(h^{-1+\sigma\varepsilon})$ as $h \rightarrow 0$.

Take \tilde{u}_2 to be the solution to $H_{W_2, V_2} \tilde{u}_2 = 0$ with $\tilde{u}_2 = u_1$ on $\partial\Omega$, and let $u = u_1 - \tilde{u}_2$. We use Lemma 2.3 for u and u_2 to get

$$(H_{W_2, V_2} u|_{u_2}) = -(\partial_\nu u|_{u_2})_{\partial\Omega}. \quad (33)$$

We examine the right hand side of (33).

Lemma 7.1. One has $h(\partial_\nu u|_{u_2})_{\partial\Omega} \rightarrow 0$ as $h \rightarrow 0$.

Proof. Because of (32) we have $\partial_\nu u = i((W_2 - W_1) \cdot \nu)u_1$ on \tilde{F} , and so

$$h(\partial_\nu u|_{u_2})_{\partial\Omega} = ih(((W_2 - W_1) \cdot \nu)u_1|_{u_2})_{\tilde{F}} + h(\partial_\nu u|_{u_2})_{\partial\Omega \setminus \tilde{F}} \quad (34)$$

The first term satisfies

$$h|(((W_2 - W_1) \cdot \nu)u_1|_{u_2})_{\tilde{F}}| \lesssim h\|a_1 + r_1\|_{L^2(\partial\Omega)}\|a_2 + r_2\|_{L^2(\partial\Omega)}.$$

We have $\|a_j\|_{L^2(\partial\Omega)} \lesssim \|a_j\|_{L^\infty(\partial\Omega)} = O(1)$ and $\|r_j\|_{L^2(\partial\Omega)} \lesssim \|r_j\|_{H^{1/2+\delta}(\Omega)} = O(h^{-1/2+\sigma\varepsilon-\delta})$ for any $\delta > 0$, which gives

$$\|a_j + r_j\|_{L^2(\partial\Omega)} = O(h^{-1/2+\delta}) \quad (35)$$

for some new $\delta > 0$. This shows that the first term in (34) vanishes as $h \rightarrow 0$.

For the other term we compute

$$|h(\partial_\nu u|u_2)_{\partial\Omega \setminus \tilde{F}}| \leq h\|e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega \setminus \tilde{F})}\|a_2 + r_2\|_{L^2(\partial\Omega)}.$$

To estimate the normal derivative we use the Carleman estimate of Proposition 4.1, for the weight $-\varphi$, in the form

$$\begin{aligned} \sqrt{h}\|\sqrt{\partial_\nu \varphi}e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega_+)} + \|e^{-\frac{\varphi}{h}}u\| + \|e^{-\frac{\varphi}{h}}h\nabla u\| \\ \lesssim h\|e^{-\frac{\varphi}{h}}H_{W_2, V_2}u\| + \sqrt{h}\|\sqrt{-\partial_\nu \varphi}e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega_-)}. \end{aligned}$$

The estimate applies for this u since $u \in H_\Delta(\Omega)$ and $u|_{\partial\Omega} = 0$. Since $e^{-\frac{\varphi}{h}}\partial_\nu u = ie^{i\frac{\psi}{h}}((W_2 - W_1) \cdot \nu)(a_1 + r_1)$ on \tilde{F} , and since $\partial\Omega_- \subseteq \tilde{F}$, it follows from (35) that

$$\begin{aligned} \|e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega \setminus \tilde{F})} &\lesssim \|\sqrt{\partial_\nu \varphi}e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega_+)} \\ &\lesssim \sqrt{h}\|e^{-\frac{\varphi}{h}}H_{W_2, V_2}u\| + O(h^{-1/2+\delta}). \end{aligned}$$

Using that u_1 and \tilde{u}_2 are solutions and $\nabla \cdot W_j = 0$, we obtain

$$H_{W_2, V_2}u = H_{W_2, V_2}u_1 = 2(W_2 - W_1) \cdot Du_1 + (W_2^2 - W_1^2 + V_2 - V_1)u_1.$$

The explicit form for u_1 implies

$$\begin{aligned} H_{W_2, V_2}u &= e^{\frac{1}{h}(\varphi+i\psi)}(2ih^{-1}[(W_1 - W_2) \cdot \nabla(\varphi + i\psi)](a_1 + r_1) \\ &\quad + 2(W_2 - W_1) \cdot D(a_1 + r_1) + (W_2^2 - W_1^2 + V_2 - V_1)(a_1 + r_1)) \end{aligned} \quad (36)$$

which shows that $\|e^{-\frac{\varphi}{h}}H_{W_2, V_2}u\| = O(h^{-1})$, since the $L^{\frac{2n}{n-2}}$ norm of $a_1 + r_1$ can be estimated by the H^1 norm. Collecting these estimates gives

$$h(\partial_\nu u|u_2)_{\partial\Omega \setminus \tilde{F}} = O(h^\delta)$$

as $h \rightarrow 0$, which concludes the proof. \square

From (36) we obtain

$$h(H_{W_2, V_2}u|u_2) = \int_\Omega 2i(W_1 - W_2) \cdot \nabla(\varphi + i\psi)a_1\bar{a}_2 dx + o(1).$$

Then (33) and Lemma 7.1 imply

$$\int_\Omega \nabla(\varphi + i\psi) \cdot (W_1 - W_2)a dx = 0 \quad (37)$$

where $a = \lim_{h \rightarrow 0} a_1\bar{a}_2$. We will now switch to the complex notation as in Section 6.

Choose coordinates so that $x_0 = 0$ and $\omega = e_1$, and write $x = (x_1, x')$ where $x' = r\theta$, $r > 0$, and $\theta \in S^{n-2}$. Let P_θ be the two-dimensional plane consisting of points $(x_1, r\theta)$ for θ fixed, and write $\Omega_\theta = \Omega \cap P_\theta$. We also use the complex variable $z = x_1 + ir$, which identifies P_θ with \mathbf{C} .

In the coordinates $(z, \theta) = \Psi(x)$, Lemma 6.1 shows that a in (37) is given by $a \circ \Psi^{-1} = (z - \bar{z})^{2-n} e^{i\Phi}$, where Φ satisfies

$$\partial_{\bar{z}} \Phi = -\frac{1}{2}(e_1 + ie_r) \cdot (\tilde{W}_1 - \tilde{W}_2) \quad \text{in } \Psi(\Omega). \quad (38)$$

We have written $\tilde{W}_j = W_j \circ \Psi^{-1}$. Note that since Φ is of the form (31), $W_j \in C_c^\varepsilon$ implies that $\Phi \in C^\varepsilon$ and further $\Phi(\cdot, \theta) \in C^{1+\varepsilon}(\mathbf{C})$. The latter follows since $\partial_{\bar{z}} \Phi \in C^\varepsilon$ and $\partial_z \Phi \in \partial_z \partial_{\bar{z}}^{-1} C_c^\varepsilon$, and the singular integral operator $\partial_z \partial_{\bar{z}}^{-1}$ is bounded on Hölder spaces.

Now (37) becomes

$$\int_{S^{n-2}} \left(\int_{\Omega_\theta} \frac{1}{z} (e_1 + ie_r) \cdot (\tilde{W}_1 - \tilde{W}_2) e^{i\Phi} d\bar{z} \wedge dz \right) d\theta = 0. \quad (39)$$

We need a slightly more general result. From the transport equation for a_1 we see that in the definition of u_1 we may replace a_1 by $a_1 g_1$, where $\nabla(\varphi + i\psi) \cdot \nabla g_1 = 0$ and $g_1 \in W^{2,\infty}(\Omega)$ is independent of h . It follows that (39) holds with $e^{i\Phi}$ replaced by $e^{i\Phi} g_1$. Choosing $g_1 = z g(z) \tilde{g}(\theta)$ with $\partial_{\bar{z}} g = 0$ and \tilde{g} smooth, and by varying \tilde{g} , we see that for almost every $\theta \in S^{n-2}$ one has

$$\int_{\Omega_\theta} (e_1 + ie_r) \cdot (\tilde{W}_1 - \tilde{W}_2) e^{i\Phi} g(z) d\bar{z} \wedge dz = 0. \quad (40)$$

The argument that (40) implies $dW_1 = dW_2$ now proceeds as in [5]. However, this was written in [5] quite briefly, especially in the case where Ω_θ has nontrivial topology. Therefore, we will give a rather detailed argument for determining the coefficients.

It will be enough to restrict to 3-dimensional subspaces L of \mathbf{R}^n which consist of the points $(x_1, r\theta)$ where $\theta \in S^{n-2}$ varies on a fixed two-plane. The fact that (40) holds in each L will imply $dW_1 = dW_2$ in Ω , see [5]. Thus we may assume $n = 3$ in what follows, and $\theta \in S^1$ may be identified with an angle, using $\angle e^{i\alpha} = \alpha$ when $0 < \alpha < \pi$.

We begin by looking at what kind of topology Ω_θ can have when θ varies. For this we use the map $\Theta : \partial\Omega \rightarrow \mathbf{R}, x \mapsto \angle \frac{x'}{|x'|}$. The main point is that possibly after a small change of coordinates, Θ is a Morse function.

Lemma 7.2. For almost every choice $(\omega, x_0) \in TS^2$, Θ is a Morse function and there are finitely many critical values $\theta_0 < \theta_1 < \dots < \theta_N$ of Θ . Also, $\Theta(\partial\Omega) = [\theta_0, \theta_N]$, and if $\theta \in (\theta_j, \theta_{j+1})$ then

- (a) $\partial\Omega$ and P_θ intersect transversally, or equivalently for any $x \in \partial\Omega \cap P_\theta$ the vector $\nu(x)$ is not orthogonal to P_θ ,
- (b) Ω_θ is a bounded open set with smooth boundary in P_θ , and $\partial\Omega_\theta = \partial\Omega \cap P_\theta$,
- (c) there is a diffeomorphism $T : (\theta_j, \theta_{j+1}) \times \partial\Omega_\theta \rightarrow \Theta^{-1}((\theta_j, \theta_{j+1}))$, and if $T_t(x) = T(t, x)$ then T_t is a diffeomorphism of $\partial\Omega_\theta$ and $\partial\Omega_t$ with $T_\theta = \text{Id}_{\partial\Omega_\theta}$.

Proof. We see that $x \in \partial\Omega$ is a degenerate critical point of Θ if and only if $N(x) \cdot \omega = N(x) \cdot (x - x_0) = 0$ and $K(x) = 0$, where N is the Gauss map and K is the Gaussian curvature of $\partial\Omega$. Thus, Θ is Morse provided that

(ω, x_0) is not a critical value of the map $F : \Omega(\partial\Omega) \times \mathbf{R} \rightarrow TS^2$,

$$F((x, \omega), t) = (\omega, (x \cdot N(x))N(x) + t(\omega \times N(x))),$$

where $\Omega(\partial\Omega) \subseteq T(\partial\Omega)$ is the unit sphere bundle. Sard's lemma shows that Θ is Morse for almost every (ω, x_0) . Then Θ has finitely many critical values, and if θ is not a critical value then $\partial\Omega$ and P_θ intersect transversally.

The other condition in (a) is just another way of stating the transversality, and this latter condition implies that $\partial\Omega_\theta = \partial\Omega \cap P_\theta$. Then, $\partial\Omega_\theta$ is a smooth 1-manifold and Ω_θ has smooth boundary in P_θ . Part (c) follows from the fact that there is a gradient-like vector field for Θ [11], and we obtain the diffeomorphism T by considering flows of this vector field which originate from $\partial\Omega_\theta$. \square

Note that $\partial\Omega_\theta$ can have many components, but if θ stays away from the critical values then the number of components stays fixed and the components vary smoothly with θ . Unless stated otherwise, we will assume that the coordinates are chosen as in Lemma 7.2 and θ is not a critical value.

Lemma 7.3. $dW_1 = dW_2$ in Ω .

Proof. Our starting point is (40). Because of the factor $e^{i\Phi}$ this may be considered as a nonlinear Radon transform, evaluated at the plane P_θ . Most of the work below is to show that (40) remains true with $e^{i\Phi}$ replaced by 1, which corresponds to the usual Radon transform.

The proof is in several steps. For the complex analysis terminology see [16].

Step 1. Using the equation (38) and integrating by parts, we obtain from (40) the orthogonality condition

$$\int_{\partial\Omega_\theta} e^{i\Phi} g(z) dz = 0 \quad (41)$$

for any holomorphic $g \in W^{2,\infty}(\Omega_\theta)$. Since $P_\theta \setminus \Omega_\theta$ has finitely many components, an approximation argument (using a version of Mergelyan's theorem) implies that (41) holds in fact for any holomorphic $g \in C(\overline{\Omega}_\theta)$.

Step 2. The condition (41) is equivalent with saying that $e^{i\Phi}|_{\partial\Omega_\theta}$ is the boundary value of a holomorphic function $F \in C(\overline{\Omega}_\theta)$. Indeed, write $f = e^{i\Phi}$, and let F be the Cauchy integral of $f|_{\partial\Omega_\theta}$. From (41) we see that $F = 0$ outside Ω_θ , and the Plemelj-Sokhotski formula then implies that $F \in C(\overline{\Omega}_\theta)$ with $F|_{\partial\Omega_\theta} = f$.

Step 3. We would like to show that F is nonvanishing and has a holomorphic logarithm in Ω_θ . We first claim that if γ is a closed curve in $\partial\Omega_\theta$, then

$$\int_{F \circ \gamma} \frac{1}{z} dz = \int_{f \circ \gamma} \frac{1}{z} dz = 0. \quad (42)$$

To show this we write $f_s = e^{is\Phi}$ for $0 \leq s \leq 1$. We obtain that $f \circ \gamma$ is homotopic to $f_0 \circ \gamma = \{1\}$ in $\mathbf{C} \setminus \{0\}$, and the claim follows.

Step 4. We can use (42) and the argument principle to conclude that F is nonvanishing, also when Ω_θ is not simply connected. Let γ be an oriented

parametrization of $\partial\Omega_\theta$ as a sum of simple closed curves. It follows that $\text{Ind}_\gamma(\alpha) = \int_{\partial\Omega_\theta} \frac{1}{z-\alpha} dz = 0$ for all $\alpha \notin \Omega_\theta$. Also, $F \neq 0$ on $\partial\Omega_\theta$, so there are only finitely many zeros in Ω_θ . One may now use the residue theorem and argue in the usual way that $\frac{1}{2\pi i} \int_{F \circ \gamma} \frac{1}{z} dz = 0$ is the number of zeros of F in Ω_θ .

Step 5. Next we will show that F has a holomorphic logarithm in Ω_θ . Again, this would be immediate in a simply connected domain. In the general case, F has a holomorphic logarithm provided that

$$\int_\gamma \frac{F'}{F} dz = 0 \quad (43)$$

for any closed curve γ in Ω_θ . To show (43) for given γ , we take C_j to be the finitely many components of $P_\theta \setminus \Omega_\theta$, and for each j we let γ_j be a cycle corresponding to the oriented boundary of C_j . Then γ_j is contained in $\partial\Omega_\theta$, the index of γ_j on C_j is 1, and the index of γ_j on any other component C_k is 0. Thus, using the γ_j , we can construct a cycle $\tilde{\gamma}$ contained in $\partial\Omega_\theta$ so that the index of $\tilde{\gamma}$ is equal to the index of γ at each point outside of Ω_θ . It follows that

$$\int_\gamma \frac{F'}{F} dz = \int_{\tilde{\gamma}} \frac{F'}{F} dz = \int_{F \circ \tilde{\gamma}} \frac{1}{z} dz = 0$$

using (42). This shows (43), and we obtain a holomorphic logarithm $G \in C(\overline{\Omega}_\theta)$ with $F = e^G$ by fixing a point z_0 in each component of Ω_θ , and by taking in this component

$$G(z) = \int_{z_0}^z \frac{F'}{F} dw + c_0 \quad (44)$$

where the integral is over any curve connecting z_0 to z , and $e^{c_0} = F(z_0)$.

Step 6. Since $e^{G-i\Phi} = 1$ on $\partial\Omega_\theta$, we get $G|_{\partial\Omega_\theta} = i\Phi + v$ where v is constant on each component of $\partial\Omega_\theta$. In fact, v is equal to $2\pi im$ for some $m \in \mathbf{Z}$ on each of these components. This shows that

$$\int_{\partial\Omega_\theta} G dz = \int_{\partial\Omega_\theta} i\Phi dz. \quad (45)$$

Step 7. We now return to (41) and take $g = \frac{G}{e^G} \in C(\overline{\Omega}_\theta)$. Then (45) gives

$$\int_{\partial\Omega_\theta} \Phi dz = 0.$$

Integrating by parts and using (38) yields

$$\int_{\Omega_\theta} (e_1 + ie_r) \cdot (\tilde{W}_1 - \tilde{W}_2) d\bar{z} \wedge dz = 0. \quad (46)$$

Step 8. We need to show that (46) is valid with e_r replaced by $-e_r$. If W_j were real valued this would follow just by taking complex conjugates. If the W_j are complex, we can go back to the beginning of this section and repeat

the construction of solutions, with ψ replaced by $-\psi$. In this way, instead of (37) we arrive at

$$\int_{\Omega} \nabla(\varphi + i\psi) \cdot (\bar{W}_1 - \bar{W}_2) a \, dx = 0$$

where $a \circ \Psi^{-1} = (z - \bar{z})^{2-n} e^{i\Phi}$, and Φ satisfies

$$\partial_{\bar{z}} \Phi = \frac{1}{2} (e_1 + i e_r) \cdot (\widetilde{W}_1 - \widetilde{W}_2) \quad \text{in } \Psi(\Omega).$$

Going through steps 1 to 8 above, we obtain (46) with W_j replaced by \bar{W}_j , and taking conjugates gives

$$\int_{\Omega_{\theta}} (e_1 - i e_r) \cdot (\tilde{W}_1 - \tilde{W}_2) \, d\bar{z} \wedge dz = 0. \quad (47)$$

Step 9. If $\xi \in P_{\theta}$ then ξ is a linear combination of e_1 and e_r , and we get from (46) and (47) that

$$\int_{\Omega_{\theta}} \xi \cdot (\tilde{W}_1 - \tilde{W}_2) \, d\bar{z} \wedge dz = 0.$$

Returning to the x coordinates, this gives

$$\int_{(x_0+P) \cap \Omega} \xi \cdot (W_1 - W_2) \, dS = 0 \quad (48)$$

for all two-planes P passing through e_1 , and all $\xi \in P$.

Step 10. The left hand side of (48) is related to the Radon transform of an expression involving $\text{curl}(W_1 - W_2)$. Varying x_0 in a small neighborhood of 0 and varying ω in a conic neighborhood of e_1 , one obtains $\text{curl}(W_1 - W_2) = 0$ from the arguments in [5, Lemma 5.2]. \square

After showing that $dW_1 = dW_2$, the final step is to show that $V_1 = V_2$. This also follows along the lines of [5], but we need to give more details to account for the nontrivial topology of the Ω_{θ} .

Lemma 7.4. $V_1 = V_2$ in Ω .

Proof. The proof is again in several steps.

Step 1. Since $\text{curl}(W_1 - W_2) = 0$ and Ω is simply connected, we have $W_2 - W_1 = \nabla p$ where $p \in C^{1+\varepsilon}(\bar{\Omega})$.

Step 2. The assumption that $N_{W_1, V_1} f|_{\tilde{F}} = N_{W_2, V_2} f|_{\tilde{F}}$ for all $f \in H^{1/2}(\partial\Omega)$, together with boundary determination results for the magnetic Schrödinger equation [1], implies that $(W_1)_{\text{tan}} = (W_2)_{\text{tan}}$ on \tilde{F} . Note that the results in [1] remain valid for $V \in L^n(\Omega)$. It follows that the tangential derivatives of p vanish on \tilde{F} , which shows that p is constant on each component of \tilde{F} .

Step 3. We use again the complex notation, with $x_0 = 0$ and $\omega = e_1$. Inserting $W_2 - W_1 = \nabla p$ in (38), we see that $\Phi = \tilde{p}$ will be a solution of (38) if $\tilde{p} = p \circ \Psi^{-1}$. Repeating the arguments in the proof of Lemma 7.3 shows that $e^{i\tilde{p}}|_{\partial\Omega_{\theta}}$ is the boundary value of a nonvanishing holomorphic function, and taking a holomorphic logarithm gives $i\tilde{p} = G_{\theta} + v_{\theta}$ on $\partial\Omega_{\theta}$,

where $G_\theta \in C(\overline{\Omega}_\theta)$ is holomorphic and v_θ is equal to some $2\pi im$, $m \in \mathbf{Z}$, on each component of $\partial\Omega_\theta$.

Step 4. We claim that for each component C_θ of Ω_θ , there is a point $y_0 \in \partial C_\theta$ with $y_0 \in \tilde{F}$. In fact, we may take y_0 to be a point which minimizes $|y|$ among $y \in \overline{C}_\theta$. Then $y_0 \in \partial C_\theta$ and $(1-t)y_0 \notin \overline{C}_\theta$ for $t > 0$. It is enough to show that $y_0 \cdot \nu(y_0) \leq 0$, since then $y_0 \in \tilde{F}$. But if one had $y_0 \cdot \nu(y_0) > 0$, then the fact that for any $c > 0$ there is a truncated cone

$$\{y \in \mathbf{R}^n : (y - y_0) \cdot \nu(y_0) < -c|y - y_0|, |y - y_0| < \delta\} \cap \Omega_\theta$$

contained in C_θ , would imply that $(1-t)y_0 \in C_\theta$ for some $t > 0$. This is a contradiction.

Step 5. From Steps 3 and 4 we see that $G_\theta \in C(\overline{\Omega}_\theta)$ is a holomorphic function in Ω_θ , and for any component C_θ of Ω_θ there is an open set of ∂C_θ in which G_θ is constant. This implies that G_θ is constant on each component of $\overline{\Omega}_\theta$, hence also on each component of $\partial\Omega_\theta$.

Step 6. We also need that $G_\theta|_{\partial\Omega_\theta}$ can be made to vary continuously with θ , as long as θ stays away from the critical values of Θ . To do this we use Lemma 7.2 (c) which implies that $\partial\Omega_\theta$ varies continuously with θ , and so does F in (44). Using the regularity properties of $\Phi = \tilde{p}$, and choosing the point z_0 in (44) to lie in $\partial\Omega_\theta$ with continuous dependence on θ , it can be checked that $G_\theta|_{\partial\Omega_\theta}$ varies continuously with θ .

Step 7. We now restrict to the set $\Theta^{-1}((\theta_j, \theta_{j+1}))$, and claim that p is locally constant in this set (which implies that p is constant on the components of this set, since p is continuous). To prove this, we first recall that $i\tilde{p} = G_\theta + v_\theta$ on $\partial\Omega_\theta$, where \tilde{p} and G_θ vary continuously with θ . Then also v_θ varies continuously with θ , and since $\frac{1}{2\pi i}v_\theta$ is integer valued we see that v_θ is locally constant.

If $x \in \Theta^{-1}((\theta_j, \theta_{j+1}))$ and $\Theta(x) = \theta$, then x lies in some component C_θ of $\overline{\Omega}_\theta$. By Step 4 there is some point y in $C_\theta \cap \tilde{F}$, and Step 2 shows that p is constant in $B(y, r) \cap \partial\Omega$ for some $r > 0$. The same then applies to G_θ near y . Since the components of $\partial\Omega_\theta$ vary continuously, and since G_θ is constant on each such component, we see that G_θ is constant near x when θ varies. This shows the claim.

Step 8. To show that p is locally constant on $\partial\Omega$, it remains to check that this is true near each $x \in \partial\Omega \cap P_\theta$ where θ is a critical value. If $\nu(x)$ is orthogonal to P_θ , then $x \in \tilde{F}$ and this follows by Step 2. If $\nu(x)$ is not orthogonal to P_θ , then for some $r > 0$ and j , both the sets $B(x, r) \cap \Theta^{-1}((\theta_j, \theta_{j+1}))$ and $B(x, r) \cap \Theta^{-1}((\theta_{j-1}, \theta_j))$ are nonempty and connected. Since p is constant on both these sets and continuous, p must be constant also near x .

Step 9. The preceding step implies that p is constant on the components of $\partial\Omega$. This shows that $(W_1)_{\text{tan}} = (W_2)_{\text{tan}}$ on $\partial\Omega$. Recalling that $\partial\Omega$ is connected, we may subtract a constant from p to obtain $p|_{\partial\Omega} = 0$. The assumption, gauge invariance, and Step 1 then imply

$$N_{W_1, V_1}|_{\tilde{F}} = N_{W_2, V_2}|_{\tilde{F}} = N_{W_2 - \nabla p, V_2}|_{\tilde{F}} = N_{W_1, V_2}|_{\tilde{F}}.$$

Consequently, we may assume $W_1 = W_2$ in the arguments in this section. Going through the proof of Lemma 7.1, and using the assumption $V_j \in L^\infty(\Omega)$, we may take limits as $h \rightarrow 0$ in (33) to obtain

$$\int_{\Omega} (V_1 - V_2) a \, dx = 0 \quad (49)$$

where $a \circ \Psi^{-1} = (z - \bar{z})^{2-n}$.

Step 10. We argue as in Lemma 7.3 and replace a in (49) by ag where $\nabla(\varphi + i\psi) \cdot \nabla g = 0$. Moving to the variables (z, θ) and taking g to be a function of θ , and by varying g , we obtain

$$\int_{\Omega_\theta} (V_1 - V_2)(\Psi^{-1}(z, \theta)) \, d\bar{z} \wedge dz = 0 \quad (50)$$

for almost every θ . Here we use the fact that $V_1 - V_2$ is in $L^1(\mathbf{R}^n)$, so the restriction $V_1 - V_2|_{P_\theta}$ is integrable on P_θ for almost every θ .

In the x coordinates, (50) reads

$$\int_{(x_0+P) \cap \Omega} (V_1 - V_2) \, dS = 0. \quad (51)$$

This is valid for almost every two-plane P passing through $\omega = e_1$. Varying x_0 and ω slightly, the Radon transform arguments in [5] give $V_1 = V_2$. \square

8. DETERMINING A CONVECTION TERM

Finally, we prove Theorem 1.2 by reducing the inverse problem for the convection equation to the corresponding problem for the magnetic Schrödinger equation.

Proof. (of Theorem 1.2) We observe that $H_{iW_j, q_j} = -\Delta + 2W_j \cdot \nabla$ where $q_j = W_j^2 - \nabla \cdot W_j$. Consequently, for $f \in H^{1/2}(\partial\Omega)$ we have the relation $N_{iW_j, q_j} f = N_{W_j} f - (W_j \cdot \nu) f$ between DN maps, and

$$(N_{iW_1, q_1} - N_{iW_2, q_2}) f|_{\tilde{F}} = -((W_1 - W_2) \cdot \nu) f|_{\tilde{F}}$$

using the assumption that $N_{W_1} f = N_{W_2} f$ on \tilde{F} . By [1], this assumption also implies that $W_1 = W_2$ on \tilde{F} , and we obtain $(N_{iW_1, q_1} - N_{iW_2, q_2}) f|_{\tilde{F}} = 0$.

Now Theorem 1.1 shows that $dW_1 = dW_2$ and $q_1 = q_2$ in Ω . Since Ω is simply connected there is $p \in C^{1+\varepsilon}(\overline{\Omega})$ with $W_2 = W_1 + \nabla p$. Theorem 1.1 also shows that $(W_1)_{\tan} = (W_2)_{\tan}$ on $\partial\Omega$, so p is constant on $\partial\Omega$ which was assumed to be connected. By subtracting the constant we may assume that $p|_{\partial\Omega} = 0$. The condition $q_1 = q_2$ then implies that

$$-\Delta p + (2W_1 + \nabla p) \cdot \nabla p = 0 \quad \text{in } \Omega.$$

Since everything is real valued, the maximum principle shows that the only solution with zero boundary values is $p \equiv 0$. It follows that $W_1 = W_2$. \square

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